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THE MIXED BOUNDARY VALUE PROBLEM FOR THE ELASTIC HALF SPACE

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Abstract—The mixed boundary problem for the half-space occupied by an elastic medium is considered. Using the Fourier transformation the problem is reduced to the Riemann problem for the system of two pairs of functions, the solution of which is obtained in quadratures.

WE CONSIDER the semi-infinite region $x_3 \geq 0$ occupied by an elastic medium, and assume that the displacements u_1 and u_2 vanish on the semi-plane $x_3 = 0$, $x_1 > 0$; the displacement u_3 is given on the whole plane $x_3 = 0$; and the tangential stresses are zero on the semiplane $x_3 = 0$, $x_1 < 0$.

The problem mathematically reduces to finding the displacements u_j $(j = 1, 2, 3)$ vanishing at infinity and satisfying Navier's equations [1]

$$\Delta u_j + \frac{m}{m-2}\frac{\partial \theta}{\partial x_j} = 0, \qquad (j = 1, 2, 3) \tag{1}$$

inside the half space; where $\theta = u_{i,i} = \operatorname{div} u$, $m = \text{const} > 2$. On the boundaries we have

$$u_j(x_1, x_2, +0) = 0; \qquad (j = 1, 2); \qquad x_1 > 0, \qquad -\infty < x_2 < +\infty \tag{2}$$

$$u_3(x_1, x_2, +0) = g(x_1, x_2), \qquad -\infty < x_1 < +\infty, \qquad -\infty < x_2 < +\infty \tag{3}$$

$$\tau_{j3}(x_1, x_2, +0) = 0, \qquad (j = 1, 2), \qquad x_1 < 0, \qquad -\infty < x_2 < +\infty. \tag{4}$$

The solution of (1) can be put in the form

$$u_j(x_1, x_2, x_3) = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} U_j(k_1, k_2; x_3)\, e^{-i(k_1x_1+k_2x_2)}\, dk_1\, dk_2, \tag{5}$$

where

$$U_j(k_1, k_2; x_3) = [A_j(k_1, k_2) + x_3 B_j(k_1, k_2)]\, e^{-\alpha x_3}; \qquad (j = 1, 2, 3), \tag{6}$$

$$\alpha = \sqrt{(k_1^2 + k_2^2)} \geq 0. \tag{7}$$

The functions $A_j(k_1, k_2)$, $B_j(k_1, k_2)$ satisfy the relations

$$B_j = \frac{-mk_j}{(3m-4)\alpha}(k_1A_1 + k_2A_2 - i\alpha A_3); \qquad (j = 1, 2); \tag{8.1}$$

$$B_3 = \frac{im}{(3m-4)}(k_1A_1 + k_2A_2 - i\alpha A_3). \tag{8.2}$$

Assume that the displacements u_1, u_2 and the stress are τ_{13}, τ_{23} on the plane $x_3 = 0$ are defined by

$$u_1(x_1, x_2, +0) = \phi_-(x_1, x_2); \qquad u_2(x_1, x_2, +0) = \psi_-(x_1, x_2); \qquad |x_1|, |x_2| < \infty \tag{9.1}$$

$$\frac{1}{\mu}\tau_{13}(x_1, x_2, +0) = \phi_+(x_1, x_2); \qquad \frac{1}{\mu}\tau_{23}(x_1, x_2, +0) = \psi_+(x_1, x_2) \tag{9.2}$$

where μ is the shear modulus and the unknown functions $\phi_\pm$, $\psi_\pm$ satisfy the conditions (2), (4) i.e.

$$\begin{aligned} \phi_+ \equiv \psi_+ \equiv 0, \qquad (x_1 < 0); \\ \phi_- \equiv \psi_- \equiv 0, \qquad (x_1 > 0). \end{aligned} \tag{10}$$

Substituting from (5) and (6) in (9) yields

$$A_1 = \Phi^-, \qquad A_2 = \Psi^-, \qquad A_3 = G \tag{11}$$

$$\begin{aligned} -ik_1A_3 + B_1 - \alpha A_1 = \Phi^+ \\ -ik_2A_3 + B_2 - \alpha A_2 = \Psi^+ \end{aligned} \tag{12}$$

where $\Phi^\pm(k_1, k_2)$; $\Psi^\pm(k_1, k_2)$ and $G(k_1, k_2)$ are the double Fourier transformations of the functions $\phi_\pm(x_1, x_2)$; $\psi_\pm(x_1, x_2)$ and $g(x_1, x_2)$ respectively.

Eliminating A_j and B_j ($j = 1, 2, 3$) from (11), (12) and (8) we arrive at the equations

$$\begin{aligned} \Phi^+ &= -\frac{1}{\alpha}\left(\alpha^2 + \frac{mk_1^2}{3m-4}\right)\Phi^- - \frac{mk_1k_2}{(3m-4)\alpha}\Psi^- - 2i\frac{m-2}{3m-4}k_1G \\ \Psi^+ &= -\frac{mk_1k_2}{(3m-4)\alpha}\Phi^- - \frac{1}{\alpha}\left(\alpha^2 + \frac{mk_2^2}{3m-4}\right)\Psi^- - 2i\frac{m-2}{3m-4}k_2G. \end{aligned} \tag{13}$$

The obtained Riemann problem (10, 13) for the system of two pairs of functions can be solved by means of simple factorization [2]. In fact taking

$$\alpha = \sqrt{(k_1^2 + k_2^2)} = -[\sqrt{(k_1 + ik_2)^+}][\sqrt{(k_1 - ik_2)^-}] = -K^+K^-$$

and then using the solution of the jump problem [2] we obtain

$$\begin{aligned} \Phi^+ &= \frac{1}{K^+}\left\{ik_1\frac{2(m-2)}{3m-4}[GK^+]^+ + C_1(k_2)\right\} \\ \Psi^+ &= \frac{1}{K^+}\left\{ik_2\frac{2(m-2)}{3m-4}[GK^+]^+ + C_2(k_2)\right\} \\ \Phi^- &= \frac{K^-}{(1+\nu)\alpha^2}[(1+\nu)k_1^2 + k_2^2][k_2\Omega^+(k_1) + C_2(k_2)] - \nu k_1k_2[k_1\Omega^+(k_1) + C_1(k_2)] \\ \Psi^- &= \frac{K^-}{(1+\nu)\alpha^2}[k_1^2 + (1+\nu)k_2^2][k_1\Omega^+(k_1) + C_1(k_2)] - \nu k_1k_2[k_2\Omega^+(k_1) + C_2(k_2)] \end{aligned} \tag{14}$$

where $C_1(k_2)$ and $C_2(k_2)$ are so far arbitrary functions, and

$$\nu = \frac{m}{3m-4}; \qquad \Omega^+(k_1) = i\frac{2(m-2)}{3m-4}[GK^+]^-. \tag{15}$$

In order that the formulae (14) and (15) would give the solution of the Riemann problem (13) it is necessary to eliminate the double poles of the functions $\Psi^-(\xi, k_2)$ and $\Phi^-(\xi, k_2)$ at the point $\xi = i|k_2|$. It turns out that it is sufficient to choose $C_1(k_2)$ and $C_2(k_2)$ in the following form

$$C_1(k_2) = -\frac{2i|k_2|\Omega^+(i|k_2|)}{v+2}; \qquad C_2(k_2) = -\frac{2k_2\Omega^+(i|k_2|)}{v+2}. \tag{16}$$

Thus, substituting from (15) and (16) into (14), then using (11) and (12) we find A_j and B_j. The solution of the problem (1–4) has the form (6).

REFERENCES

[1] I. S. Sokolnikoff, *Mathematical Theory of Elasticity*. McGraw-Hill (1956).
[2] B. Noble, *Methods Based on the Wiener–Hopf Technique*. Pergamon Press (1958).
[3] F. D. Gakhov, *Boundary Value Problems*. Pergamon Press (1966).

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Абстракт—Исследуется смешанная краевая задача для полупространства занимаемого упругой средой. Путем применения преобразования Фурье, задача сводится к задаче Риманна для системы двух пар функций, решение которых получается в квадратурах.